Bäcklund transformations on surfaces with $(k_1 - m)(k_2 - m) = \pm l^2$ in $R^{2,1}$

1997 J. Phys. A: Math. Gen. 30 6009

(http://iopscience.iop.org/0305-4470/30/17/012)

J. Phys. A: Math. Gen. **30** (1997) 6009–6020. Printed in the UK PII: S0305-4470(97)78084-3

Bäcklund transformations on surfaces with $(k_1 - m)(k_2 - m) = \pm l^2$ in $R^{2,1}$

Xi-Fang Cao and Chou Tian

Department of Mathematics, University of Science and Technology of China, Hefei 230026, People's Republic of China

Received 24 September 1996, in final form 1 April 1997

Abstract. In this paper, we obtain three types of Bäcklund transformations on surfaces with $(k_1 - m)(k_2 - m) = \pm l^2$ in three-dimensional Minkowski space $R^{2,1}$, where k_1 and k_2 are principal curvatures, $m(\geqslant 0)$ and $l(> 0)$ are arbitrary constants. Using these transformations, one can construct families of surfaces with $(k_1 - m)(k_2 - m) = \pm l^2$ in $R^{2,1}$ from known ones.

1. Introduction

A Weingarten surface is that for which a functional relationship exists between the principal curvatures k_1 and k_2. For example, minimal surfaces ($k_1 = -k_2$), constant mean curvature surfaces ($k_1 + k_2 =$ constant), surfaces all of whose points are umbilics ($k_1 = k_2$) and boundaries of tubes ($k_1 =$ constant) are all Weingarten surfaces. The history of Weingarten surfaces goes back to the last century. It was named after J Weingarten, who made the first study in 1861 [1]. After that, there is a wealth of interesting results about Weingarten surfaces from Hilbert, Chern, Hopf, Voss, and many others [2]. For example, in 1945 Chern [3] proved that a compact convex special Weingarten surface is a sphere. More recently, Huang [4] and Wu [5] discussed the relationship between Weingarten surfaces and differential equations.

Constant curvature surfaces ($k_1 k_2 =$ constant) are also Weingarten surfaces. It is well known that there exists the Bäcklund transformation on negative constant curvature surfaces in three-dimensional Euclidean space R^3 [6]. Using this transformation, one can construct a family of negative constant curvature surfaces from a known one. With the development of the soliton theory, Bäcklund transformation has attracted interest from many mathematicians. In 1981, Chern [7] gave a connection between positive constant curvature surfaces in $R^{2,1}$ and the sine-Gordon and sinh-Gordon equations. In 1982, Hu [8] reported a connection between negative constant curvature surfaces in $R^{2,1}$ and the sine-Laplace and sinh-Laplace equations. Based on the works of Chern [7] and Hu [8], Huang [9] obtained three types of Bäcklund transformations on constant curvature surfaces in $R^{2,1}$, and therefore generalized the Bäcklund transformation on negative constant curvature surfaces in R^3 to the constant curvature surfaces in $R^{2,1}$. Recently, Tian [10, 11] generalized the Bäcklund transformation on negative constant curvature surfaces in R^3 to the Weingarten surfaces with the condition $(k_1 - m)(k_2 - m) = -l^2$ in R^3, where k_1 and k_2 are principal curvatures, $m(\geqslant 0)$ and $(l(> 0)$ are arbitrary constants. In this paper, we obtain three types of Bäcklund transformations are arbitrary constants. In this paper, we obtain three types of Bäcklund transformations on Weingarten surfaces with the condition $(k_1 - m)(k_2 - m) = \pm l^2$ in $R^{2,1}$ i.e.

0305-4470/97/176009+12\$19.50

(1) Bäcklund transformation between spacelike surfaces with $(k_1-m)(k_2-m)=l^2$ and timelike surfaces with $(k_1-m)(k_2-m)=-l^2$;

(2) Bäcklund transformation on spacelike surfaces with $(k_1-m)(k_2-m)=-l^2$;

(3) Bäcklund transformation on timelike surfaces with $(k_1-m)(k_2-m)=l^2$, therefore generalize the results of [9] to the Weingarten surfaces $(k_1-m)(k_2-m)=\pm l^2$ in $R^{2,1}$.

In section 2 of this paper, we give the Bäcklund transformation (1) in detail. Because the proofs of (2) and (3) are similar to that of (1), we only give the main results of (2) and (3) in sections 3 and 4 respectively, and omit all the proofs.

We suppose that the surfaces discussed in this paper do not contain any umbilic points.

2. Bäcklund transformation between spacelike surfaces with $(k_1-m)(k_2-m)=l^2$ and timelike surfaces with $(k_1-m)(k_2-m)=-l^2$

Let S be a spacelike surface with $(k_1-m)(k_2-m)=l^2$ $(m^2-l^2)\neq 0$. Let $\{r; e_1, e_2, e_3\}$ be an orthonormal Lorentzian frame field on S such that e_1, e_2 are tangent vector fields in the principal directions, e_3 is the normal vector field ($e_1^2=e_2^2=-e_3^2=1$). Then we have the moving equations

$$\mathrm{d}r=\omega_1 e_1+\omega_2 e_2 \qquad \mathrm{d}e_i=\sum_{j=1}^{3}\omega_{ij}e_j \qquad i=1,2,3$$

where $\omega_{12}=-\omega_{21}$, $\omega_{13}=\omega_{31}$, $\omega_{23}=\omega_{32}$.

Suppose the first and second fundamental forms of S are respectively

$$I=a^2\,\mathrm{d}u^2+b^2\,\mathrm{d}v^2 \qquad II=k_1a^2\,\mathrm{d}u^2+k_2b^2\,\mathrm{d}v^2$$

then we have

$$\begin{aligned}
&\omega_1=a\,\mathrm{d}u \qquad \omega_2=b\,\mathrm{d}v\\
&\omega_{13}=\omega_{31}=-k_1a\,\mathrm{d}u\\
&\omega_{23}=\omega_{32}=-k_2b\,\mathrm{d}v\\
&\omega_{12}=-\omega_{21}=-\frac{a_v}{b}\,\mathrm{d}u+\frac{b_u}{a}\,\mathrm{d}v
\end{aligned}$$

and the Codazzi equations

$$(k_1-k_2)a_v+k_{1v}a=0 \qquad (k_2-k_1)b_u+k_{2u}b=0.$$

Let

$$k_1-m=l\,\mathrm{ch}\,\frac{\alpha}{2}\bigg/\mathrm{sh}\,\frac{\alpha}{2} \qquad k_2-m=l\,\mathrm{sh}\,\frac{\alpha}{2}\bigg/\mathrm{ch}\,\frac{\alpha}{2}$$

then by the Codazzi equations, we can choose parameters u and v such that

$$a=\mathrm{sh}\,\frac{\alpha}{2} \qquad b=\mathrm{ch}\,\frac{\alpha}{2}.$$

Hence

$$\begin{cases}
\omega_1=\mathrm{sh}\,\frac{\alpha}{2}\,\mathrm{d}u \qquad \omega_2=\mathrm{ch}\,\frac{\alpha}{2}\,\mathrm{d}v\\
\omega_{12}=-\omega_{21}=-\frac{\alpha_v}{2}\,\mathrm{d}u+\frac{\alpha_u}{2}\,\mathrm{d}v\\
\omega_{13}=\omega_{31}=-\left(l\,\mathrm{ch}\,\frac{\alpha}{2}+m\,\mathrm{sh}\,\frac{\alpha}{2}\right)\mathrm{d}u\\
\omega_{23}=\omega_{32}=-\left(l\,\mathrm{sh}\,\frac{\alpha}{2}+m\,\mathrm{ch}\,\frac{\alpha}{2}\right)\mathrm{d}v
\end{cases} \tag{1}$$

and the Gauss equation is

$$\alpha_{uu} + \alpha_{vv} = (m^2 + l^2)\,\mathrm{sh}\,\alpha + 2ml\,\mathrm{ch}\,\alpha \tag{2}$$

(the Codazzi equations are identities).

Therefore we have

Lemma 2.1. Let S be a spacelike surface with $(k_1 - m)(k_2 - m) = l^2(m^2 - l^2 \neq 0)$, then S can be covered by charts with coordinate (u, v) such that the first fundamental form is

$$I = \mathrm{sh}^2\frac{\alpha}{2}\,\mathrm{d}u^2 + \mathrm{ch}^2\frac{\alpha}{2}\,\mathrm{d}v^2$$

and the second fundamental form is

$$II = \mathrm{sh}\frac{\alpha}{2}\left(l\,\mathrm{ch}\frac{\alpha}{2} + m\,\mathrm{sh}\frac{\alpha}{2}\right)\mathrm{d}u^2 + \mathrm{ch}\frac{\alpha}{2}\left(l\,\mathrm{sh}\frac{\alpha}{2} + m\,\mathrm{ch}\frac{\alpha}{2}\right)\mathrm{d}v^2,$$

where $\alpha(u, v)$ satisfies the equation

$$\alpha_{uu} + \alpha_{vv} = (m^2 + l^2)\,\mathrm{sh}\,\alpha + 2ml\,\mathrm{ch}\,\alpha.$$

Lemma 2.2. If $\alpha(u, v)$ satisfies equation (2), $\tau(\neq 0)$ is an arbitrary constant, then the following equations on α'

$$\frac{1}{2}\mathrm{ch}\,\tau(\alpha'_v + \alpha_u) = \cos\frac{\alpha'}{2}\left(l\,\mathrm{ch}\frac{\alpha}{2} + m\,\mathrm{sh}\frac{\alpha}{2}\right) - \mathrm{sh}\,\tau\sin\frac{\alpha'}{2}\left(l\,\mathrm{sh}\frac{\alpha}{2} + m\,\mathrm{ch}\frac{\alpha}{2}\right) \tag{3}$$

$$\frac{1}{2}\mathrm{ch}\,\tau(\alpha'_u - \alpha_v) = -\sin\frac{\alpha'}{2}\left(l\,\mathrm{sh}\frac{\alpha}{2} + m\,\mathrm{ch}\frac{\alpha}{2}\right) - \mathrm{sh}\,\tau\cos\frac{\alpha'}{2}\left(l\,\mathrm{ch}\frac{\alpha}{2} + m\,\mathrm{sh}\frac{\alpha}{2}\right) \tag{4}$$

are completely integrable, and $\alpha'(u, v)$ satisfies the equation

$$\alpha'_{uu} + \alpha'_{vv} = (m^2 - l^2)\sin\alpha'. \tag{5}$$

Proof. Since (3) and (4),

$$\begin{aligned}\frac{1}{2}\mathrm{ch}\,\tau(\alpha'_{vu} + \alpha_{uu}) = {}& -\frac{\alpha'_u}{2}\sin\frac{\alpha'}{2}\left(l\,\mathrm{ch}\frac{\alpha}{2} + m\,\mathrm{sh}\frac{\alpha}{2}\right) + \frac{\alpha_u}{2}\cos\frac{\alpha'}{2}\left(l\,\mathrm{sh}\frac{\alpha}{2} + m\,\mathrm{ch}\frac{\alpha}{2}\right)\\ & -\frac{\alpha'_u}{2}\mathrm{sh}\,\tau\cos\frac{\alpha'}{2}\left(l\,\mathrm{sh}\frac{\alpha}{2} + m\,\mathrm{ch}\frac{\alpha}{2}\right) - \frac{\alpha_u}{2}\mathrm{sh}\,\tau\sin\frac{\alpha'}{2}\left(l\,\mathrm{ch}\frac{\alpha}{2} + m\,\mathrm{sh}\frac{\alpha}{2}\right)\\ \frac{1}{2}\mathrm{ch}\,\tau(\alpha'_{uv} - \alpha_{vv}) = {}& -\frac{\alpha'_v}{2}\cos\frac{\alpha'}{2}\left(l\,\mathrm{sh}\frac{\alpha}{2} + m\,\mathrm{ch}\frac{\alpha}{2}\right) - \frac{\alpha_v}{2}\sin\frac{\alpha'}{2}\left(l\,\mathrm{ch}\frac{\alpha}{2} + m\,\mathrm{sh}\frac{\alpha}{2}\right)\\ & +\frac{\alpha'_v}{2}\mathrm{sh}\,\tau\sin\frac{\alpha'}{2}\left(l\,\mathrm{ch}\frac{\alpha}{2} + m\,\mathrm{sh}\frac{\alpha}{2}\right) - \frac{\alpha_v}{2}\mathrm{sh}\,\tau\cos\frac{\alpha'}{2}\left(l\,\mathrm{sh}\frac{\alpha}{2} + m\,\mathrm{ch}\frac{\alpha}{2}\right)\end{aligned}$$

then

$$\begin{aligned}\frac{1}{2}\mathrm{ch}^2\,\tau(\alpha'_{vu} - \alpha'_{uv} + \alpha_{uu} + \alpha_{vu}) = {}& \frac{1}{2}\mathrm{ch}\,\tau(\alpha'_u - \alpha_v)\Bigg[-\sin\frac{\alpha'}{2}\left(l\,\mathrm{ch}\frac{\alpha}{2} + m\,\mathrm{sh}\frac{\alpha}{2}\right)\\ & -\mathrm{sh}\,\tau\cos\frac{\alpha'}{2}\left(l\,\mathrm{sh}\frac{\alpha}{2} + m\,\mathrm{ch}\frac{\alpha}{2}\right)\Bigg]\\ & +\frac{1}{2}\mathrm{ch}\,\tau(\alpha'_v + \alpha_u)\Bigg[\cos\frac{\alpha'}{2}\left(l\,\mathrm{sh}\frac{\alpha}{2} + m\,\mathrm{ch}\frac{\alpha}{2}\right)\\ & -\mathrm{sh}\,\tau\sin\frac{\alpha'}{2}\left(l\,\mathrm{ch}\frac{\alpha}{2} + m\,\mathrm{sh}\frac{\alpha}{2}\right)\Bigg].\end{aligned} \tag{6}$$

Substituting (3) and (4) into (6), we obtain

$$\alpha'_{vu} - \alpha'_{uv} + \alpha_{uu} + \alpha_{vv} = (m^2 + l^2)\,\mathrm{sh}\,\alpha + 2ml\,\mathrm{ch}\,\alpha.$$

Since $\alpha(u,v)$ satisfies (2), (3) and (4) are completely integrable. Since (3) and (4),

$$\frac{1}{2}\operatorname{ch}\tau(\alpha'_{vv}+\alpha_{uv}) = -\frac{\alpha'_v}{2}\sin\frac{\alpha'}{2}\left(l\operatorname{ch}\frac{\alpha}{2}+m\operatorname{sh}\frac{\alpha}{2}\right)+\frac{\alpha_v}{2}\cos\frac{\alpha'}{2}\left(l\operatorname{sh}\frac{\alpha}{2}+m\operatorname{ch}\frac{\alpha}{2}\right)$$
$$-\frac{\alpha'_v}{2}\operatorname{sh}\tau\cos\frac{\alpha'}{2}\left(l\operatorname{sh}\frac{\alpha}{2}+m\operatorname{ch}\frac{\alpha}{2}\right)-\frac{\alpha_v}{2}\operatorname{sh}\tau\sin\frac{\alpha'}{2}\left(l\operatorname{ch}\frac{\alpha}{2}+m\operatorname{sh}\frac{\alpha}{2}\right)$$
$$\frac{1}{2}\operatorname{ch}\tau(\alpha'_{uu}-\alpha_{vu}) = -\frac{\alpha'_u}{2}\cos\frac{\alpha'}{2}\left(l\operatorname{sh}\frac{\alpha}{2}+m\operatorname{ch}\frac{\alpha}{2}\right)-\frac{\alpha_u}{2}\sin\frac{\alpha'}{2}\left(l\operatorname{ch}\frac{\alpha}{2}+m\operatorname{sh}\frac{\alpha}{2}\right)$$
$$+\frac{\alpha'_u}{2}\operatorname{sh}\tau\sin\frac{\alpha'}{2}\left(l\operatorname{ch}\frac{\alpha}{2}+m\operatorname{sh}\frac{\alpha}{2}\right)-\frac{\alpha_u}{2}\operatorname{sh}\tau\cos\frac{\alpha'}{2}\left(l\operatorname{sh}\frac{\alpha}{2}+m\operatorname{ch}\frac{\alpha}{2}\right)$$

then

$$\frac{1}{2}\operatorname{ch}^2\tau(\alpha'_{uu}+\alpha'_{vv}) = \frac{1}{2}\operatorname{ch}\tau(\alpha'_v+\alpha_u)\Bigg[-\sin\frac{\alpha'}{2}\left(l\operatorname{ch}\frac{\alpha}{2}+m\operatorname{sh}\frac{\alpha}{2}\right)$$
$$-\operatorname{sh}\tau\cos\frac{\alpha'}{2}\left(l\operatorname{sh}\frac{\alpha}{2}+m\operatorname{ch}\frac{\alpha}{2}\right)\Bigg]$$
$$+\frac{1}{2}\operatorname{ch}\tau(\alpha'_u-\alpha_v)\Bigg[-\cos\frac{\alpha'}{2}\left(l\operatorname{sh}\frac{\alpha}{2}+m\operatorname{ch}\frac{\alpha}{2}\right)$$
$$+\operatorname{sh}\tau\sin\frac{\alpha'}{2}\left(l\operatorname{ch}\frac{\alpha}{2}+m\operatorname{sh}\frac{\alpha}{2}\right)\Bigg]. \tag{7}$$

Substituting (3) and (4) into (7), we have

$$\alpha'_{uu}+\alpha'_{vv} = (m^2-l^2)\sin\alpha'.$$

Therefore, (3) and (4) give the Bäcklund transformation between (2) and (5). □

Let

$$e = \cos\frac{\alpha'}{2}e_1+\sin\frac{\alpha'}{2}e_2$$
$$e^{\perp} = -\sin\frac{\alpha'}{2}e_1+\cos\frac{\alpha'}{2}e_2$$
$$e'_3 = \operatorname{ch}\tau e^{\perp}+\operatorname{sh}\tau e_3$$

where $\alpha(u,v)$ is a solution of the equations (3) and (4).

Suppose S' is a surface defined by

$$r' = \frac{m^2-l^2}{m^2+l^2}r+\frac{\operatorname{ch}\tau}{m^2+l^2}(le-me^{\perp})+\frac{m}{m^2+l^2}(1-\operatorname{sh}\tau)e_3. \tag{8}$$

Theorem 2.1. e'_3 is the normal vector field of S'.

Proof. Since (1),

$$\mathrm{d}r = \operatorname{sh}\frac{\alpha}{2}\,\mathrm{d}u e_1+\operatorname{ch}\frac{\alpha}{2}\,\mathrm{d}v e_2,$$
$$\mathrm{d}e = \left[\frac{1}{2}(\alpha'_u-\alpha_v)\,\mathrm{d}u+\frac{1}{2}(\alpha'_v+\alpha_u)\,\mathrm{d}v\right]e^{\perp}$$
$$-\left[\cos\frac{\alpha'}{2}\left(l\operatorname{ch}\frac{\alpha}{2}+m\operatorname{sh}\frac{\alpha}{2}\right)\mathrm{d}u+\sin\frac{\alpha'}{2}\left(l\operatorname{sh}\frac{\alpha}{2}+m\operatorname{ch}\frac{\alpha}{2}\right)\mathrm{d}v\right]e_3$$
$$\mathrm{d}e^{\perp} = -\left[\frac{1}{2}(\alpha'_u-\alpha_v)\,\mathrm{d}u+\frac{1}{2}(\alpha'_v+\alpha_u)\,\mathrm{d}v\right]e$$

$$+\left[\sin\frac{\alpha'}{2}\left(l\,\mathrm{ch}\,\frac{\alpha}{2}+m\,\mathrm{sh}\,\frac{\alpha}{2}\right)\mathrm{d}u-\cos\frac{\alpha'}{2}\left(l\,\mathrm{sh}\,\frac{\alpha}{2}+m\,\mathrm{ch}\,\frac{\alpha}{2}\right)\mathrm{d}v\right]e_3 \tag{9}$$

$$\mathrm{d}e_3=-\left(l\,\mathrm{ch}\,\frac{\alpha}{2}+m\,\mathrm{sh}\,\frac{\alpha}{2}\right)\mathrm{d}u e_1-\left(l\,\mathrm{sh}\,\frac{\alpha}{2}+m\,\mathrm{ch}\,\frac{\alpha}{2}\right)\mathrm{d}v e_2 \tag{10}$$

then

$$\frac{m^2-l^2}{m^2+l^2}\,\mathrm{d}r e_3'=\frac{m^2-l^2}{m^2+l^2}\,\mathrm{ch}\,\tau\left(-\sin\frac{\alpha'}{2}\,\mathrm{sh}\,\frac{\alpha}{2}\,\mathrm{d}u+\cos\frac{\alpha'}{2}\,\mathrm{ch}\,\frac{\alpha}{2}\,\mathrm{d}v\right)$$

$$\frac{l\,\mathrm{ch}\,\tau}{m^2+l^2}\,\mathrm{d}e e_3'=\frac{l}{m^2+l^2}\,\mathrm{ch}^2\,\tau\left[\frac{1}{2}(\alpha_u'-\alpha_v)\,\mathrm{d}u+\frac{1}{2}(\alpha_v'+\alpha_u)\,\mathrm{d}v\right]$$
$$+\frac{l\,\mathrm{ch}\,\tau\,\mathrm{sh}\,\tau}{m^2+l^2}\left[\cos\frac{\alpha'}{2}\left(l\,\mathrm{ch}\,\frac{\alpha}{2}+m\,\mathrm{sh}\,\frac{\alpha}{2}\right)\mathrm{d}u+\sin\frac{\alpha'}{2}\left(l\,\mathrm{sh}\,\frac{\alpha}{2}+m\,\mathrm{ch}\,\frac{\alpha}{2}\right)\mathrm{d}v\right]$$

$$-\frac{m\,\mathrm{ch}\,\tau}{m^2+l^2}\,\mathrm{d}e^{\perp}e_3'=\frac{m\,\mathrm{ch}\,\tau\,\mathrm{sh}\,\tau}{m^2+l^2}\left[\sin\frac{\alpha'}{2}\left(l\,\mathrm{ch}\,\frac{\alpha}{2}+m\,\mathrm{sh}\,\frac{\alpha}{2}\right)\mathrm{d}u\right.$$
$$\left.-\cos\frac{\alpha'}{2}\left(l\,\mathrm{sh}\,\frac{\alpha}{2}+m\,\mathrm{ch}\,\frac{\alpha}{2}\right)\mathrm{d}v\right]$$

$$\frac{m(1-\mathrm{sh}\,\tau)}{m^2+l^2}\,\mathrm{d}e_3 e_3'=\frac{m(1-\mathrm{sh}\,\tau)\,\mathrm{ch}\,\tau}{m^2+l^2}\left[\sin\frac{\alpha'}{2}\left(l\,\mathrm{ch}\,\frac{\alpha}{2}+m\,\mathrm{sh}\,\frac{\alpha}{2}\right)\mathrm{d}u\right.$$
$$\left.-\cos\frac{\alpha'}{2}\left(l\,\mathrm{sh}\,\frac{\alpha}{2}+m\,\mathrm{ch}\,\frac{\alpha}{2}\right)\mathrm{d}v\right].$$

By using (3) and (4), we can check directly that

$$\mathrm{d}r' e_3'=0,$$

i.e. e_3' is the normal vector field of S'.

The theorem is proved. □

Since $e_3'^2=1$, S' is a timelike surface.

Lemma 2.3. For the surface S', the first fundamental form is

$$I'=\frac{m^2-l^2}{(m^2+l^2)^2}\left[\left(l\cos\frac{\alpha'}{2}+m\sin\frac{\alpha'}{2}\right)^2\mathrm{d}u^2-\left(l\sin\frac{\alpha'}{2}-m\cos\frac{\alpha'}{2}\right)^2\mathrm{d}v^2\right]. \tag{11}$$

Proof. Since (8),

$$\mathrm{d}r'=\frac{m^2-l^2}{m^2+l^2}\,\mathrm{d}r+\frac{m}{m^2+l^2}(1-\mathrm{sh}\,\tau)\,\mathrm{d}e_3+\frac{\mathrm{ch}\,\tau}{m^2+l^2}(l\,\mathrm{d}e-m\,\mathrm{d}e^{\perp})$$
$$=a_1+a_2,$$

where

$$a_1=\frac{m^2-l^2}{m^2+l^2}\left[\omega_1+\frac{m(1-\mathrm{sh}\,\tau)}{m^2-l^2}\omega_{13}\right]e_1+\frac{m^2-l^2}{m^2+l^2}\left[\omega_2+\frac{m(1-\mathrm{sh}\,\tau)}{m^2-l^2}\omega_{23}\right]e_2 \tag{12}$$

$$a_2=\frac{\mathrm{ch}\,\tau}{m^2+l^2}\left(\frac{\mathrm{d}\alpha'}{2}+\omega_{12}\right)(le^{\perp}+me)$$
$$+\frac{\mathrm{ch}\,\tau}{m^2+l^2}\left[\left(l\cos\frac{\alpha'}{2}+m\sin\frac{\alpha'}{2}\right)\omega_{13}+\left(l\sin\frac{\alpha'}{2}-m\cos\frac{\alpha'}{2}\right)\omega_{23}\right]e_3. \tag{13}$$

Rewriting (3) and (4) as

$$\operatorname{ch}\tau\left(\frac{d\alpha'}{2}+\omega_{12}\right)=\frac{m^2-l^2}{l}\sin\frac{\alpha'}{2}\left(\omega_1+\frac{m}{m^2-l^2}\omega_{13}\right)+\operatorname{sh}\tau\cos\frac{\alpha'}{2}\omega_{13}$$
$$-\frac{m^2-l^2}{l}\cos\frac{\alpha'}{2}\left(\omega_2+\frac{m}{m^2-l^2}\omega_{23}\right)+\operatorname{sh}\tau\sin\frac{\alpha'}{2}\omega_{23}. \tag{14}$$

Then

$$I'=dr'\,dr'=a_1^2+a_2^2+2a_1a_2 \tag{15}$$

where

$$a_1^2=\frac{(m^2-l^2)^2}{(m^2+l^2)^2}\Bigg[\omega_1^2+\omega_2^2+\frac{m^2(1-\operatorname{sh}\tau)^2}{(m^2-l^2)^2}(\omega_{13}^2+\omega_{23}^2)$$
$$+\frac{2m(1-\operatorname{sh}\tau)}{m^2-l^2}(\omega_1\omega_{13}+\omega_2\omega_{23})\Bigg] \tag{16}$$

$$a_2^2=\frac{\operatorname{ch}^2\tau}{m^2+l^2}\left(\frac{d\alpha'}{2}+\omega_{12}\right)^2-\frac{\operatorname{ch}^2\tau}{(m^2+l^2)^2}\Bigg[\left(l\cos\frac{\alpha'}{2}+m\sin\frac{\alpha'}{2}\right)\omega_{13}$$
$$+\left(l\sin\frac{\alpha'}{2}-m\cos\frac{\alpha'}{2}\omega_{23}\right)\Bigg]^2 \tag{17}$$

$$2a_1a_2=\frac{2(m^2-l^2)}{(m^2+l^2)^2}\left(-l\sin\frac{\alpha'}{2}+m\cos\frac{\alpha'}{2}\right)\operatorname{ch}\tau\left(\frac{d\alpha'}{2}+\omega_{12}\right)\left[\omega_1+\frac{m(1-\operatorname{sh}\tau)}{m^2-l^2}\omega_{13}\right]$$
$$+\frac{2(m^2-l^2)}{(m^2+l^2)^2}\left(l\cos\frac{\alpha'}{2}+m\sin\frac{\alpha'}{2}\right)\operatorname{ch}\tau\left(\frac{d\alpha'}{2}+\omega_{12}\right)$$
$$\times\left[\omega_2+\frac{m(1-\operatorname{sh}\tau)}{m^2-l^2}\omega_{23}\right]. \tag{18}$$

Substituting (1) and (14) into (15)–(18) and note that

$$\omega_1+\frac{m}{m^2-l^2}\omega_{13}=-\frac{l}{m^2-l^2}\left(l\operatorname{sh}\frac{\alpha}{2}+m\operatorname{ch}\frac{\alpha}{2}\right)du \tag{19}$$

$$\omega_2+\frac{m}{m^2-l^2}\omega_{23}=-\frac{l}{m^2-l^2}\left(l\operatorname{ch}\frac{\alpha}{2}+m\operatorname{sh}\frac{\alpha}{2}\right)dv \tag{20}$$

we can check directly that (11) is established. □

Lemma 2.4. For the surface S', the second fundamental form is

$$II'=\frac{m^2-l^2}{m^2+l^2}\left[\sin\frac{\alpha'}{2}\left(l\cos\frac{\alpha'}{2}+m\sin\frac{\alpha'}{2}\right)du^2+\cos\frac{\alpha'}{2}\left(l\sin\frac{\alpha'}{2}-m\cos\frac{\alpha'}{2}\right)dv^2\right]. \tag{21}$$

Proof. Since

$$dr'=a_1+a_2 \qquad de_3'=b_1+b_2$$

where

$$b_1=\operatorname{ch}\tau\,de^{\perp} \qquad b_2=\operatorname{sh}\tau\,de_3$$

we have

$$II'=-dr'\,de_3'=-(a_1b_1+a_1b_2+a_2b_1+a_2b_2). \tag{22}$$

From (9), (10), (12) and (13), we have

$$a_1b_1 = -\frac{m^2-l^2}{m^2+l^2}\cos\frac{\alpha'}{2}\,\mathrm{ch}\,\tau\left(\frac{\mathrm{d}\alpha'}{2}+\omega_{12}\right)\left[\omega_1+\frac{m(1-\mathrm{sh}\,\tau)}{m^2-l^2}\omega_{13}\right]$$
$$-\frac{m^2-l^2}{m^2+l^2}\sin\frac{\alpha'}{2}\,\mathrm{ch}\,\tau\left(\frac{\mathrm{d}\alpha'}{2}+\omega_{12}\right)\left[\omega_2+\frac{m(1-\mathrm{sh}\,\tau)}{m^2-l^2}\omega_{23}\right] \tag{23}$$

$$a_1b_2 = \frac{m^2-l^2}{m^2+l^2}\,\mathrm{sh}\,\tau\,\omega_{13}\left[\omega_1+\frac{m(l-\mathrm{sh}\,\tau)}{m^2-l^2}\omega_{13}\right]$$
$$+\frac{m^2-l^2}{m^2+l^2}\,\mathrm{sh}\,\tau\,\omega_{23}\left[\omega_2+\frac{m(1-\mathrm{sh}\,\tau)}{m^2-l^2}\omega_{23}\right] \tag{24}$$

$$a_2b_1 = -\frac{m}{m^2+l^2}\,\mathrm{ch}^2\,\tau\left(\frac{\mathrm{d}\alpha'}{2}+\omega_{12}\right)^2$$
$$-\frac{\mathrm{ch}^2\,\tau}{m^2+l^2}\left(\cos\frac{\alpha'}{2}\omega_{23}-\sin\frac{\alpha'}{2}\omega_{13}\right)\left(l\cos\frac{\alpha'}{2}+m\sin\frac{\alpha'}{2}\right)\omega_{13}$$
$$-\frac{\mathrm{ch}^2\,\tau}{m^2+l^2}\left(\cos\frac{\alpha'}{2}\omega_{23}-\sin\frac{\alpha'}{2}\omega_{13}\right)\left(l\sin\frac{\alpha'}{2}-m\cos\frac{\alpha'}{2}\right)\omega_{23} \tag{25}$$

$$a_2b_2 = \frac{\mathrm{sh}\,\tau}{m^2+l^2}\left(-l\sin\frac{\alpha'}{2}+m\cos\frac{\alpha'}{2}\right)\mathrm{ch}\,\tau\left(\frac{\mathrm{d}\alpha'}{2}+\omega_{12}\right)\omega_{13}$$
$$+\frac{\mathrm{sh}\,\tau}{m^2+l^2}\left(l\cos\frac{\alpha'}{2}+m\sin\frac{\alpha'}{2}\right)\mathrm{ch}\,\tau\left(\frac{\mathrm{d}\alpha'}{2}+\omega_{12}\right)\omega_{13}. \tag{26}$$

Substituting (1) and (14) into (22)–(26), and by using (19) and (20), we can check directly that (21) is established. □

Theorem 2.2. The principal curvatures k_1' and k_2' of S' satisfy

$$(k_1'-m)(k_2'-m) = -l^2.$$

Proof. Since Lemmas (2.3) and (2.4),

$$k_1' = (m^2+l^2)\sin\frac{\alpha'}{2}\bigg/\left(l\cos\frac{\alpha'}{2}+m\sin\frac{\alpha'}{2}\right)$$
$$k_2' = -(m^2+l^2)\cos\frac{\alpha'}{2}\bigg/\left(l\sin\frac{\alpha'}{2}-m\cos\frac{\alpha'}{2}\right)$$

hence

$$(k_1'-m)(k_2'-m) = -l^2.$$

The theorem is proved. □

From the above discussion, to construct a family of timelike surfaces with $(k_1-m)(k_2-m) = -l^2$ from a known spacelike surface with $(k_1-m)(k_2-m) = l^2$, we only need to solve the completely integrable equations (3) and (4).

Inversely, suppose S' is a timelike surface with (11) and (21) as its first and second fundamental forms respectively, then its principal curvatures k_1' and k_2' satisfy $(k_1'-m)(k_2'-m) = -l^2$ (we may assume $m^2-l^2 > 0$). Let $\{r'; e_1', e_2', e_3'\}$ be an orthonormal Lorentzian frame field on S' such that e_1', e_2' are tangent vector fields in the principal directions, e_3' is the normal vector field ($e_1'^2 = -e_2'^2 = e_3'^2 = 1$). Then we have the moving equations

$$\mathrm{d}r' = w_1'e_1' + w_2'e_2' \qquad \mathrm{d}e_i' = \sum_{j=1}^{3} w_{ij}'e_j' \qquad i = 1, 2, 3.$$

where

$$w_1' = \frac{\sqrt{m^2-l^2}}{m^2+l^2}\left(l\cos\frac{\alpha'}{2} + m\sin\frac{\alpha'}{2}\right)\mathrm{d}u$$
$$w_2' = \frac{\sqrt{m^2-l^2}}{m^2+l^2}\left(l\sin\frac{\alpha'}{2} - m\cos\frac{\alpha'}{2}\right)\mathrm{d}v$$
$$w_{12}' = \omega_{21}' = -\frac{\alpha_v'}{2}\,\mathrm{d}u + \frac{\alpha_u'}{2}\,\mathrm{d}v$$
$$w_{13}' = -w_{31}' = \sqrt{m^2-l^2}\sin\frac{\alpha}{2}\,\mathrm{d}u$$
$$w_{23}' = w_{32}' = \sqrt{m^2-l^2}\cos\frac{\alpha}{2}\,\mathrm{d}v$$

and $\alpha'(u, v)$ satisfies equation (5).

Let

$$e' = -\frac{l\,\mathrm{sh}\,\frac{\alpha}{2} + m\,\mathrm{ch}\,\frac{\alpha}{2}}{\sqrt{m^2-l^2}}e_1' + \frac{l\,\mathrm{ch}\,\frac{\alpha}{2} + m\,\mathrm{sh}\,\frac{\alpha}{2}}{\sqrt{m^2-l^2}}e_2'$$
$$e'^{\perp} = \frac{l\,\mathrm{ch}\,\frac{\alpha}{2} + m\,\mathrm{sh}\,\frac{\alpha}{2}}{\sqrt{m^2-l^2}}e_1' - \frac{l\,\mathrm{sh}\,\frac{\alpha}{2} + m\,\mathrm{ch}\,\frac{\alpha}{2}}{\sqrt{m^2-l^2}}e_2'$$
$$e_3 = \mathrm{ch}\,\tau e'^{\perp} - \mathrm{sh}\,\tau e_3'$$

where $\alpha(u, v)$ is a solution of the equations (3) and (4).

Suppose S is a surface defined by

$$r = \frac{m^2+l^2}{m^2-l^2}r' - \frac{\mathrm{ch}\,\tau}{m^2-l^2}(le' + me'^{\perp}) + \frac{m}{m^2-l^2}(1+\mathrm{sh}\,\tau)e_3'.$$

For the surface S, we can also prove the following results.

Theorem 2.1′. e_3 is the normal vector field of S. Since $e_3^2 = -1$, S is a spacelike surface.

Theorem 2.2′. The principal curvatures k_1 and k_2 of S satisfy

$$(k_1 - m)(k_2 - m) = l^2.$$

Therefore, we obtain the Bäcklund transformation between spacelike surfaces with $(k_1 - m)(k_2 - m) = l^2$ and timelike surfaces with $(k_1 - m)(k_2 - m) = -l^2$.

Example. For the spacelike surface S:

$$r = \left(\frac{m}{m^2-l^2}\,\mathrm{sh}\sqrt{m^2-l^2}v,\ \frac{l}{\sqrt{m^2-l^2}}u,\ \frac{m}{m^2-l^2}\,\mathrm{ch}\sqrt{m^2-l^2}v\right) \qquad (27)$$

where we assume $m^2 - l^2 > 0$, we have

$$e_1 = (0, 1, 0)$$
$$e_2 = (\mathrm{ch}\,\theta, 0, \mathrm{sh}\,\theta)$$
$$e_3 = (-\,\mathrm{sh}\,\theta, 0, -\,\mathrm{ch}\,\theta)$$

where

$$\theta = \sqrt{m^2-l^2}v.$$

The first and second fundamental forms of S are respectively

$$I = \frac{l^2}{m^2-l^2}\,\mathrm{d}u^2 + \frac{m^2}{m^2-l^2}\,\mathrm{d}v^2 \qquad \text{and} \qquad II = m\,\mathrm{d}v^2.$$

The principal curvatures of S are

$$k_1 = 0 \qquad k_2 = m - \frac{l^2}{m}$$

hence

$$(k_1 - m)(k_2 - m) = l^2.$$

$$\operatorname{sh}\frac{\alpha}{2} = -\frac{l}{\sqrt{m^2 - l^2}} \qquad \operatorname{ch}\frac{\alpha}{2} = \frac{m}{\sqrt{m^2 - l^2}}.$$

Equations (3) and (4) are reduced to

$$\begin{cases} \frac{1}{2}\operatorname{ch}\tau\,\alpha'_v = -\sqrt{m^2 - l^2}\operatorname{sh}\tau\sin\frac{\alpha'}{2} \\ \frac{1}{2}\operatorname{ch}\tau\,\alpha'_u = -\sqrt{m^2 - l^2}\sin\frac{\alpha'}{2} \end{cases}$$

then we have

$$\operatorname{tg}\frac{\alpha'}{4} = e^w \qquad w = -\frac{\sqrt{m^2 - l^2}}{\operatorname{ch}\tau}u - \frac{\operatorname{sh}\tau}{\operatorname{ch}\tau}\sqrt{m^2 - l^2}\,v + c$$

where c is an arbitrary constant.

Furthermore

$$\sin\frac{\alpha'}{2} = \operatorname{sech} w \qquad \cos\frac{\alpha'}{2} = -\operatorname{th} w$$

$$e = \cos\frac{\alpha'}{2}e_1 + \sin\frac{\alpha'}{2}e_2 = (\operatorname{sech} w\operatorname{ch}\theta, -\operatorname{th} w, \operatorname{sech} w\operatorname{sh}\theta)$$

$$e^\perp = -\sin\frac{\alpha'}{2}e_1 + \cos\frac{\alpha'}{2}e_2 = (-\operatorname{th} w\operatorname{ch}\theta, -\operatorname{sech} w, -\operatorname{th} w\operatorname{sh}\theta)$$

and

$$\begin{aligned} r' &= \frac{m^2 - l^2}{m^2 + l^2}r + \frac{\operatorname{ch}\tau}{m^2 + l^2}(le - me^\perp) + \frac{m}{m^2 + l^2}(1 - \operatorname{sh}\tau)e_3 \\ &= \left(\frac{m\operatorname{sh}\tau\operatorname{sh}\theta}{m^2 + l^2} + \frac{\operatorname{ch}\tau\operatorname{ch}\theta}{m^2 + l^2}(l\operatorname{sech} w + m\operatorname{th} w),\right. \\ &\quad \frac{l\sqrt{m^2 - l^2}}{m^2 + l^2}u + \frac{\operatorname{ch}\tau}{m^2 + l^2}(-l\operatorname{th} w + m\operatorname{sech} w), \\ &\quad \left.\frac{m\operatorname{sh}\tau\operatorname{ch}\theta}{m^2 + l^2} + \frac{\operatorname{ch}\tau\operatorname{sh}\theta}{m^2 + l^2}(l\operatorname{sech} w + m\operatorname{th} w)\right). \end{aligned} \tag{28}$$

Therefore we obtain a family (depend on τ and c) of timelike surfaces (28) with $(k_1 - m)(k_2 - m) = -l^2$ from a spacelike surface (27) with $(k_1 - m)(k_2 - m) = l^2$.

3. Bäcklund transformation on spacelike surfaces with $(k_1 - m)(k_2 - m) = -l^2$

Let S be a spacelike surface with $(k_1 - m)(k_2 - m) = -l^2$. Let $\{r; e_1, e_2, e_3\}$ be an orthonormal Lorentzian frame field on S such that e_1, e_2 are tangent vector fields in the principal directions, e_3 is the normal vector field ($e_1^2 = e_2^2 = -e_3^2 = 1$). Then we can choose parameters u and v such that

$$\omega_1 = \cos\frac{\alpha}{2}\,\mathrm{d}u \qquad \omega_2 = \sin\frac{\alpha}{2}\,\mathrm{d}v$$

$$\omega_{12} = -\omega_{21} = \frac{\alpha_v}{2}\,\mathrm{d}u + \frac{\alpha_u}{2}\,\mathrm{d}v$$
$$\omega_{13} = \omega_{31} = -\left(l\sin\frac{\alpha}{2} + m\cos\frac{\alpha}{2}\right)\mathrm{d}u$$
$$\omega_{23} = \omega_{32} = \left(l\cos\frac{\alpha}{2} - m\sin\frac{\alpha}{2}\right)\mathrm{d}v$$

where $\alpha(u, v)$ satisfies

$$\alpha_{uu} - \alpha_{vv} = (m^2 - l^2)\sin\alpha - 2ml\cos\alpha. \tag{29}$$

The following equations

$$\frac{1}{2}\,\mathrm{sh}\,\tau(\alpha'_u + \alpha_v) = -\sin\frac{\alpha'}{2}\left(l\cos\frac{\alpha}{2} - m\sin\frac{\alpha}{2}\right) - \mathrm{ch}\,\tau\cos\frac{\alpha'}{2}\left(l\sin\frac{\alpha}{2} + m\cos\frac{\alpha}{2}\right) \tag{30}$$
$$\frac{1}{2}\,\mathrm{sh}\,\tau(\alpha'_v + \alpha_u) = \cos\frac{\alpha'}{2}\left(l\sin\frac{\alpha}{2} + m\cos\frac{\alpha}{2}\right) + \mathrm{ch}\,\tau\sin\frac{\alpha'}{2}\left(l\cos\frac{\alpha}{2} - m\sin\frac{\alpha}{2}\right) \tag{31}$$

give the Bäcklund transformation between equation (29) and the equation

$$\alpha'_{uu} - \alpha'_{vv} = -(m^2 + l^2)\sin\alpha'. \tag{32}$$

Let

$$e = \cos\frac{\alpha'}{2}e_1 + \sin\frac{\alpha}{2}e_2$$
$$e^{\perp} = -\sin\frac{\alpha'}{2}e_1 + \cos\frac{\alpha'}{2}e_2$$
$$e'_3 = \mathrm{ch}\,\tau e_3 + \mathrm{sh}\,\tau e^{\perp}$$

where $\alpha'(u, v)$ is a solution of the equations (30) and (31).

Suppose S' is a surface defined by

$$r' = r - \frac{\mathrm{sh}\,\tau}{m^2 + l^2}(le + me^{\perp}) + \frac{m}{m^2 + l^2}(1 - \mathrm{ch}\,\tau)e_3$$

then we have the following theorems.

Theorem 3.1. e'_3 is the normal vector field of S'. Since $e'^2_3 = -1$, S' is a spacelike surface.

Theorem 3.2. The principal curvature k'_1 and k'_2 of S' satisfy

$$(k'_1 - m)(k'_2 - m) = -l^2$$

as well.

Therefore, we give the Bäcklund transformation on spacelike surfaces with $(k_1 - m)(k_2 - m) = -l^2$.

4. Bäcklund transformation on timelike surfaces with $(k_1 - m)(k_2 - m) = l^2$

Let S be a timelike surface, which has real principal curvatures k_1 and k_2 with $(k_1 - m)(k_2 - m) = l^2 (m^2 - l^2 \neq 0)$. Let $\{r; e_1, e_2, e_3\}$ be an orthonormal Lorentzian frame field on S such that e_1, e_2 are tangent vector fields in the principal directions, e_3 is the normal vector field $(e_1^2 = -e_2^2 = e_3^2 = 1)$. Then we can choose parameters u and v such that

$$\omega_1 = \mathrm{sh}\,\frac{\alpha}{2}\,\mathrm{d}u \qquad \omega_2 = \mathrm{ch}\,\frac{\alpha}{2}\,\mathrm{d}v$$
$$\omega_{12} = \omega_{21} = \frac{\alpha_v}{2}\,\mathrm{d}u + \frac{\alpha_u}{2}\,\mathrm{d}v$$

$$\omega_{13} = -\omega_{31} = \left(l \operatorname{ch}\frac{\alpha}{2} + m \operatorname{sh}\frac{\alpha}{2}\right) \mathrm{d}u$$

$$\omega_{23} = \omega_{32} = -\left(l \operatorname{sh}\frac{\alpha}{2} + m \operatorname{ch}\frac{\alpha}{2}\right) \mathrm{d}v$$

where $\alpha(u, v)$ satisfies

$$\alpha_{uu} - \alpha_{vv} = -(m^2 + l^2) \operatorname{sh}\alpha - 2ml \operatorname{ch}\alpha. \tag{33}$$

The following equations

$$\frac{1}{2} \operatorname{sh}\tau(\alpha'_u + \alpha_v) = -\operatorname{sh}\frac{\alpha'}{2}\left(l \operatorname{sh}\frac{\alpha}{2} + m \operatorname{ch}\frac{\alpha}{2}\right) + \operatorname{ch}\tau \operatorname{ch}\frac{\alpha'}{2}\left(l \operatorname{ch}\frac{\alpha}{2} + m \operatorname{sh}\frac{\alpha}{2}\right) \tag{34}$$

$$\frac{1}{2} \operatorname{sh}\tau(\alpha'_v + \alpha_u) = \operatorname{ch}\frac{\alpha'}{2}\left(l \operatorname{ch}\frac{\alpha}{2} + m \operatorname{sh}\frac{\alpha}{2}\right) - \operatorname{ch}\tau \operatorname{sh}\frac{\alpha'}{2}\left(l \operatorname{sh}\frac{\alpha}{2} + m \operatorname{ch}\frac{\alpha}{2}\right) \tag{35}$$

give the Bäcklund transformation between equation (33) and the equation

$$\alpha'_{uu} - \alpha'_{vv} = -(m^2 - l^2) \operatorname{sh}\alpha'. \tag{36}$$

Let

$$e = \operatorname{ch}\frac{\alpha'}{2} e_1 + \operatorname{sh}\frac{\alpha'}{2} e_2$$

$$e^{\perp} = \operatorname{sh}\frac{\alpha'}{2} e_1 + \operatorname{ch}\frac{\alpha'}{2} e_2$$

$$e'_3 = -\operatorname{ch}\tau e_3 - \operatorname{sh}\tau e^{\perp}$$

where $\alpha'(u, v)$ is a solution of the equations (34) and (35).

Suppose S' is a surface defined by

$$r' = r + \frac{\operatorname{sh}\tau}{m^2 - l^2}(le + me^{\perp}) + \frac{m}{m^2 - l^2}(1 + \operatorname{ch}\tau)e_3$$

then we have the following theorems.

Theorem 4.1. e'_3 is the normal vector field of S'. Since $e'^2_3 = 1$, S' is a timelike surface.

Theorem 4.2. The principal curvatures k'_1 and k'_2 of S' satisfy

$$(k'_1 - m)(k'_2 - m) = l^2$$

as well.

Therefore, we give the Bäcklund transformation on timelike surfaces with $(k_1 - m)(k_2 - m) = l^2$.

Acknowledgments

The work was supported by the Fund of Education Ministry of China and Academy Sinica.

References

[1] Weingarten J 1861 Über eine Klasse auf einander abwickelbaren Flächen *J. Reine Angew. Math.* **59** 382–93
[2] Hopf H 1951 Über Flächen mit einer Relation zwischen der Haptkrümmungen *Math. Nachr.* **4** 232–49
[3] Chern S S 1945 Some new characterizations of Euclidean spheres *Dukl. Math. J.* **12** 279–90
[4] Huang X G 1988 Weingarten surfaces in three-dimensional spaces *Acta. Math. Sinica* **31** 332–40 (in Chinese)
[5] Wu H 1993 Weingarten surfaces and nonlinear partial differential equations *Ann. Glob. Anal. Geom.* **11** 49–64
[6] Eisenhart L P 1909 *A Treatise in Differential Geometry of Curves and Surfaces* (New York: Ginn)
[7] Chern S S 1981 Geometrical interpretation of sinh-Gordon equation *Ann. Pol. Math.* **39** 63–9

[8] Hu H S 1985 The construction of hyperbolic surfaces in 3-dimensional Minkowski space and sinh-Laplace equation *Acta Math. Sinica* **1** 79–86
[9] Huang Y Z 1986 Bäcklund theorem on constant curvature surfaces in 3-dimensional Minkowski space and its higher dimension generalization *Acta Math. Sinica* **29** 684–90 (in Chinese)
[10] Tian C 1995 A generalization of Bäcklund theorem *Chinese Science Bulletin* **40** 1854–60
[11] Tian C 1995 Bäcklund transformation on surfaces with $(k_1 - m)(k_2 - m) = -l^2$ *Preprint* IC/95/296, ICTP (Trieste, Italy)